Poisson process, Markov processes and some queueing networks

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Outline

- Poisson processes
- Markov jump processes
- Some queueing networks

The Poisson distribution (Siméon-Denis Poisson, 1781-1840)

$\left\{e^{-\lambda}\frac{\lambda^n}{n!}\right\}_{n\in\mathbb{N}}$ As prevalent as Gaussian distribution

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Law of *rare events* (a.k.a. *law of small numbers*)
$p_{n,i} \geq 0$ such that $\lim_{n\to\infty} \sup_i p_{n,i} = 0$, $\lim_{n\to\infty} \sum_i p_{n,i} = \lambda > 0$

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Then $X_n = \sum_i Z_{n,i}$ with $Z_{n,i}$: independent Bernoulli($p_{n,i}$) verifies

$$X_n \xrightarrow{\mathcal{D}} \text{Poisson}(\lambda)$$

Point process on $\mathbb{R}_+$

Definition

Point process on $\mathbb{R}_+$:

Collection of random times $\{T_n\}_{n>0}$ with $0 < T_1 < T_2 \dots$

Alternative description
Collection $\{N_t\}_{t\in\mathbb{R}_+}$ with $N_t := \sum_{n>0} \mathbb{I}_{T_n\in[0,t]}$

Yet another description
Collection $\{N(C)\}$ for all measurable $C \subset \mathbb{R}_+$ where

$$N(C) := \sum_{n>0} \mathbb{I}_{T_n\in C}$$

Poisson process on $\mathbb{R}_+$

Definition

Point process such that for all $s_0 = 0 < s_1 < s_2 < \ldots < s_n$,

1. Increments $\{N_{s_i} - N_{s_{i-1}}\}_{1 \le i \le n}$ independent
2. Law of $N_{t+s} - N_s$ only depends on t
3. for some $\lambda > 0$, $N_t \sim \text{Poisson}(\lambda t)$

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λ is called the **intensity** of the process

A first construction

Given $\lceil \lambda n \rceil$ i.i.d. numbers $U_{n,i}$, uniform on $[0, n]$, let $N_t^{(n)} := \sum_{i=1}^{n} \mathbb{I}_{U_{n,i} \le t}$

Then for any $k \in \mathbb{N}$, $s_0 = 0 < s_1 < s_2 < \ldots < s_n$,

$$\{N_{s_i}^{(n)} - N_{s_{i-1}}^{(n)}\}_{1 \le i \le k} \stackrel{\mathcal{D}}{\to} \otimes_{1 \le i \le k} \text{Poisson}(\lambda(s_i - s_{i-1}))$$

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Proof: Multinomial distribution of $\{N_{s_i}^{(n)} - N_{s_{i-1}}^{(n)}\}_{1 \le i \le k}$

$\Rightarrow$ Convergence of Laplace transform $\mathbb{E} \exp(-\sum_{i=1}^k \alpha_i (N_{s_i}^{(n)} - N_{s_{i-1}}^{(n)}))$ for all $\alpha_1^k \in \mathbb{R}_+^k$

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Suggests Poisson processes exist and are limits of this construction

Structure of interarrival sequences

Proposition

For Poisson process $\{T_n\}_{n>0}$ of intensity λ, its interarrival times $\tau_i = T_{i+1} - T_i$, where $T_0 = 0$, verify
$\{\tau_n\}_{n\geq 0}$ i.i.d. with common distribution $\text{Exp}(\lambda)$

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Key property: Exponential random variable τ is **memoryless**, i.e. $\forall t > 0,\ \mathbb{P}(\tau - t \in \cdot | \tau > t) = \mathbb{P}(\tau \in \cdot)$

Alternative construction

Proposition

Process with i.i.d., $\text{Exp}(\lambda)$ interarrival times $\{\tau_i\}_{i\geq 0}$ can be constructed on $[0, t]$ by
1) Drawing $N_t \sim \text{Poisson}(\lambda t)$
2) Putting N_t points $U_1, \ldots, U_{N_t}$ on $[0, t]$ where U_i: i.i.d. uniform on $[0, t]$

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Proof.

Establish identity for all $n \in \mathbb{N}$, $\phi : \mathbb{R}_+^n \to \mathbb{R}$:

$$\mathbb{E}[\phi(\tau_0, \tau_0+\tau_1, \ldots, \tau_0+\ldots+\tau_{n-1})\mathbb{I}_{N_t=n}] = \cdots$$

$$e^{-\lambda t}\frac{(\lambda t)^n}{n!} \times n! \int_{(0,t]^n} \phi(s_1, s_2, \ldots, s_n)\mathbb{I}_{s_1<s_2<\ldots<s_n} \prod_{i=1}^n \frac{1}{t} ds_i$$

$$= \mathbb{P}(\text{Poisson}(\lambda t) = n) \times \mathbb{E}[\phi(S_1, \ldots, S_n)]$$

where S_1^n: sorted version of i.i.d. variables uniform on $[0,t]$ □

Laplace transform of Poisson processes

Definition

For function $f : \mathbb{R}_+ \to \mathbb{R}_+$, let

$$N(f) := \sum_{n>0} f(T_n).$$

The Laplace transform of point process $N \leftrightarrow \{T_n\}_{n>0}$ is the functional whose evaluation at $f : \mathbb{R}_+ \to \mathbb{R}_+$ is

$$\mathcal{L}_N(f) := \mathbb{E} \exp(-N(f)) = \mathbb{E}(\exp(-\sum_{n>0} f(T_n))).$$

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Proposition: Knowledge of $\mathcal{L}_N(f)$ on sufficiently rich class of functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ characterizes law of point process N.
Exemples of such classes of functions:
- piecewise constant with compact support
- piecewise continuous with compact support
- continuous with compact support

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$$\mathcal{L}_N(f) = \exp(-\int_{\mathbb{R}_+} \lambda(1 - e^{-f(x)})dx)$$

(i) Previous construction yields expression for $\mathcal{L}_N(f)$
(ii) For $f = \sum_i \alpha_i \mathbb{I}_{C_i} \Rightarrow N(C_i) \sim \text{Poisson}(\lambda \int_{C_i} dx)$, with independence for disjoint C_i. Hence existence of Poisson process...

Poisson process with general space and intensity

Definition

For $\lambda : \mathbb{R}^d \to \mathbb{R}_+$ locally integrable function, $N \leftrightarrow \{T_n\}_{n>0}$ point process on $\mathbb{R}^d$ is Poisson with intensity function λ if and only if for measurable, disjoint $C_i \subset \mathbb{R}^d, i = 1, \ldots, n$,
$N(C_i)$ independent, $\sim$ Poisson$(\int_{C_i} \lambda(x)dx)$

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Proposition

Such a process exists and admits Laplace transform

$$\mathcal{L}_N(f) = \exp\left(-\int_{\mathbb{R}^d} \lambda(x)(1 - e^{-f(x)})dx\right)$$

Further properties

- **Strong Markov property:** Poisson process N with intensity λ, stopping time T (i.e. $\forall t \geq 0$, $\{T \leq t\} \in \sigma(N_s, s \leq t)$) then on $\{T < +\infty\}$, $\{N_{T+t} - N_T\}_{t \geq 0}$: Poisson with intensity λ and independent of $\{N_s\}_{s \leq T}$

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- **Superposition:** For independent Poisson processes N_i with intensities λ_i, $i = 1, \ldots, n$ then $N = \sum_i N_i$: Poisson with intensity $\sum_i \lambda_i$

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- **Thinning**: For Poisson process $N \leftrightarrow \{T_n\}_{n>0}$ with intensity λ, $\{Z_n\}_{n>0}$ independent of N, i.i.d., valued in $[k]$, processes $N_i : N_i(C) = \sum_{n>0} \mathbb{I}_{T_n \in C} \mathbb{I}_{Z_n = i}$ are independent, Poisson with intensities $\lambda_i = \lambda \mathbb{P}(Z_n = i)$

Markov jump processes

Process $\{X_t\}_{t \in \mathbb{R}_+}$ with values in E, countable or finite, is

Markov if
$\mathbb{P}(X_{t_n} = x_n | X_{t_{n-1}} = x_{n-1}, \ldots X_{t_1} = x_1) = \mathbb{P}(X_{t_n} = x_n | X_{t_{n-1}} = x_{n-1})$,
$t_1^n \in \mathbb{R}_+^n, \; t_1 < \cdots < t_n, \; x_1^n \in E^n$

Homogeneous if $\mathbb{P}(X_{t+s} = y | X_s = x) =: p_{xy}(t)$ independent of s,
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$\Rightarrow$ Semi-group property $p_{xy}(t+s) = \sum_{z\in E} p_{xz}(t) p_{zy}(s)$,
or $P(t+s) = P(t)P(s)$ with $P(t) = \{p_{xy}(t)\}_{x,y\in E}$

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Definition

$\{X_t\}_{t\in\mathbb{R}_+}$ is a **pure jump** Markov process if in addition
(i) It spends with probability 1 a strictly positive time in each state
(ii) Trajectories $t \to X_t$ are right-continuous

Markov jump processes: examples

- Poisson process $\{N_t\}_{t\in\mathbb{R}_+}$: then Markov jump process with $p_{xy}(t) = \mathbb{P}(\text{Poisson}(\lambda t) = y - x)$

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- Single-server queue, FIFO ("First-in-first-out") discipline, arrival times: N Poisson (λ), service times: i.i.d. Exp(μ) independent of N
  X_t= number of customers present at time t: Markov jump process by Memoryless property of Exponential distribution + Markov property of Poisson process
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- Infinite server queue with Poisson arrivals and Exponential service times: customer arrived at T_n stays in system till $T_n + \sigma_n$, where σ_n: service time
  $X_t=$ number of customers present at time t:
  Markov jump process (the $M/M/\infty/\infty$ queue)

Structure of Markov jump processes

Infinitesimal Generator

$\forall x, y,\ y \neq x \in E$, limits $q_x := \lim_{t\to 0} \frac{1-p_{xx}(t)}{t}$, $q_{xy} = \lim_{t\to 0} \frac{p_{xy}(t)}{t}$ exist in $\mathbb{R}_+$ and satisfy $\sum_{y\neq x} q_{xy} = q_x$

q_{xy}: **Jump rate** from x to y

$Q := \{q_{xy}\}_{x,y\in E}$ where $q_{xx} = -q_x$: **Infinitesimal Generator** of process $\{X_t\}_{t\in\mathbb{R}_+}$

Formally: $Q = \lim_{h\to 0} \frac{1}{h}[P(h) - I]$ where I: identity matrix

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Structure of Markov jump processes

Sequence $\{Y_n\}_{n\in\mathbb{N}}$ of visited states: Markov chain with transition matrix $p_{xy} = \mathbb{I}_{x\neq y}\frac{q_{xy}}{q_x}$

Conditionally on $\{Y_n\}_{n\in\mathbb{N}}$, sojourn times $\{\tau_n\}_{n\in\mathbb{N}}$ in successive states Y_n: independent, with distributions $\mathrm{Exp}(q_{Y_n})$

Examples

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- For FIFO $M/M/1/\infty$ queue, non-zero rates: $q_{x,x+1} = \lambda$, $q_{x,x-1} = \mu \mathbb{I}_{x>0},\ x \in \mathbb{N}$ hence $q_x = \lambda + \mu \mathbb{I}_{x>0}$

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Structure of Markov jump processes (continued)

Let $T_n := \sum_{k=0}^{n-1} \tau_k$: time of n-th jump.

If $T_\infty = +\infty$ almost surely: trajectory determined on $\mathbb{R}_+$, hence generator Q determines law of process $\{X_t\}_{t \in \mathbb{R}_+}$

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Sufficient conditions for non-explosiveness:

- $\sup_{x\in E} q_x < +\infty$
- Recurrence of induced chain $\{Y_n\}_{n\in\mathbb{N}}$
- For **Birth and Death** processes (i.e. $E = \mathbb{N}$, only non-zero rates: $\beta_n = q_{n,n+1}$, birth rate; $\delta_n = q_{n,n-1}$, death rate), non-explosiveness holds if

$$\sum_{n>0} \frac{1}{\beta_n + \delta_n} = +\infty$$

Kolmogorov's forward and backward equations

Formal differentiation of $P(t+h) = P(t)P(h) = P(h)P(t)$ yields

$\frac{d}{dt}P(t) = P(t)Q$ — Kolmogorov's forward equation

$\frac{d}{dt}p_{xy}(t) = \sum_{z\in E} p_{xz}(t)q_{zy}$

$\frac{d}{dt}P(t) = QP(t)$ — Kolmogorov's backward equation

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Follow directly from $Q = \lim_{h \to 0} \frac{1}{h}[P(h) - I]$ for finite E, in which case $P(t) = \exp(tQ), \; t \geq 0$

Hold more generally–in particular for non-explosive processes–with a more involved proof (justifying exchange of summation and differentiation)

Stationary distributions and measures

Definition

Measure $\{\pi_x\}_{x \in E}$ is **stationary** if it satisfies $\pi^T Q = 0$, or equivalently the **global balance equations**

$$\forall x \in E, \; \underbrace{\pi_x \sum_{y \neq x} q_{xy}}_{\text{flow out of } x} = \underbrace{\sum_{y \neq x} \pi_y q_{yx}}_{\text{flow into } x}$$

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EXAMPLE: stationarity for birth and death processes

$$\begin{aligned} \pi_0\beta_0 &= \pi_1\delta_1, \\ \pi_x(\beta_x+\delta_x) &= \pi_{x-1}\beta_{x-1} + \pi_{x+1}\delta_{x+1},\ x \geq 1 \end{aligned}$$

Irreducibility, recurrence, invariance

Definition

- Process $\{X_t\}_{t\in\mathbb{R}_+}$ is **irreducible** (respectively, **irreducible recurrent**) if induced chain $\{Y_n\}_{n\in\mathbb{N}}$ is.

- State x is **positive recurrent** if $\mathbb{E}_x(R_x) < +\infty$, where
$$R_x = \inf\{t > \tau_0 : X_t = x\}.$$

- Measure π is **invariant** for process $\{X_t\}_{t\in\mathbb{R}_+}$ if for all $t > 0$, $\pi^T P(t) = \pi^T$, i.e.
$$\forall x \in E, \sum_{y\in E} \pi_y p_{yx}(t) = \pi_x.$$

Limit theorems 1

Theorem

For irreducible recurrent $\{X_t\}_{t\in\mathbb{R}_+}$, $\exists$ *invariant measure* π, *unique up to some scalar factor. It can be defined as, for any* $x \in E$*:*

$$\forall y \in E,\ \pi_y = \mathbb{E}_x \int_0^{R_x} \mathbb{I}_{X_t=y} dt,$$

or alternatively with $T_x := \inf\{n > 0 : Y_n = x\}$,

$$\forall y \in E,\ \pi_y = \frac{1}{q_y} \mathbb{E}_x \sum_{n=1}^{T_x} \mathbb{I}_{Y_n=y}.$$

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$$\forall y \in E, \ \pi_y = \frac{1}{q_y} \mathbb{E}_x \sum_{n=1}^{T_x} \mathbb{I}_{Y_n=y}.$$

COROLLARIES

- $\{\hat{\pi}_y\}$ invariant for $\{Y_n\}_{n\in\mathbb{N}} \Leftrightarrow \{\hat{\pi}_y/q_y\}$ invariant for $\{X_t\}_{t\in\mathbb{R}_+}$. Thus invariance $\Leftrightarrow$ stationarity.
- For irreducible recurrent $\{X_t\}_{t\in\mathbb{R}_+}$, either all or no state $x \in E$ is positive recurrent.

Limit theorems 2

Theorem

$\{X_t\}_{t\in\mathbb{R}_+}$ *is* **ergodic** *(i.e. irreducible, positive recurrent) iff it is irreducible, non-explosive and such that* $\exists\pi$, *probability distribution satisfying global balance equations.*
Then π *is also the unique invariant probability distribution.*

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Theorem

For ergodic $\{X_t\}_{t\in\mathbb{R}_+}$ *with stationary distribution* π, *any initial distribution for* X_0 *and* π*-integrable* f,

$$\text{almost surely } \lim_{t\to\infty}\frac{1}{t}\int_0^t f(X_s)ds = \sum_{x\in E}\pi_x f(x) \quad \text{(ergodic theorem)}$$

and in distribution $X_t \xrightarrow{\mathcal{D}} \pi$ *as* $t\to\infty$.

Limit theorems 3

Theorem

For irreducible, non-ergodic $\{X_t\}_{t\in\mathbb{R}_+}$*, any initial distribution for* X_0*, then for all* $x \in E$*,*

$$\lim_{t\to\infty} \mathbb{P}(X_t = x) = 0.$$

Time reversal and reversibility

For stationary ergodic $\{X_t\}_{t\in\mathbb{R}}$ with stationary distribution π, time-reversed process $\tilde{X}_t = X_{-t}$:
Markov with transition rates $\tilde{q}_{xy} = \frac{\pi_y q_{yx}}{\pi_x}$

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Definition

Stationary ergodic $\{X_t\}_{t\in\mathbb{R}}$ with stationary distribution π **reversible** iff distributed as time-reversal $\{\tilde{X}_t\}_{t\in\mathbb{R}}$, i.e.

$$\forall x \neq y \in E, \qquad \underbrace{\pi_x q_{xy}}_{\text{flow from } x \text{ to } y} = \underbrace{\pi_y q_{yx}}_{\text{flow from } y \text{ to } x},$$

detailed balance equations.

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Detailed balance, i.e. reversibility for π implies global balance for π.
EXAMPLE: for birth and death processes, detailed balance always holds for stationary measure.

Reversibility and truncation

Proposition

Let generator Q on E admit reversible measure π. Then for subset $F \subset E$, **truncated generator** $\hat{Q}$:

$$\begin{aligned} \hat{Q}_{xy} &= Q_{xy},\ x \neq y \in F, \\ \hat{Q}_{xx} &= -\sum_{y \neq x} \hat{Q}_{xy},\ x \in F \end{aligned}$$

admits $\{\pi_x\}_{x \in F}$ as reversible measure.

Erlang's model of telephone network

- Call types $s \in \mathcal{S}$: type-s calls arrive at instants of Poisson (λ_s) process, last (if accepted) for duration Exponential (μ_s)
- type-s calls require one circuit (unit of capacity) per link $\ell \in s$
- Link ℓ has capacity C_ℓ circuits

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Stationary probability distribution:

$$\pi_x = \frac{1}{Z} \prod_{s \in \mathcal{S}} \frac{\rho_s^{x_s}}{x_s!} \prod_{\ell} \mathbb{I}_{\sum_{s \ni \ell} x_s \leq C_\ell},$$

where: $\rho_s = \lambda_s / \mu_s$, Z: normalizing constant.

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Basis for dimensioning studies of telephone networks (prediction of call rejection probabilities)
More recent application: performance analysis of peer-to-peer systems for video streaming.

Jackson networks

- Stations $i \in I$ receive external arrivals at Poisson rate $\overline{\lambda}_i$
- Station i when processing x_i customers completes service at rate $\mu_i \phi_i(x_i)$ (e.g.: $\phi_i(x) = \min(x_i, n_i)$: queue with n_i servers and service times Exponential (μ_i))
- After completing service at station i, customer joins station j with probability $p_{ij}, j \in I$, and leaves system with probability $1 - \sum_{j \in I} p_{ij}$
- Matrix $P = (p_{ij})$: sub-stochastic, such that $\exists (I - P)^{-1}$

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TRAFFIC EQUATIONS

$$\forall i \in I,\ \lambda_i = \overline{\lambda}_i + \sum_{j\in I} \lambda_j p_{ji}$$

or $\lambda = (I - P^T)^{-1}\overline{\lambda}$

Jackson networks (continued)

Stationary measure:

$$\pi_x = \prod_{i \in I} \frac{\rho_i^{x_i}}{\prod_{m=1}^{x_i} \phi_i(m)},$$

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Application: process ergodic when π has finite mass. e.g. for $\phi_i(x) = \min(x, n_i)$, ergodicity iff $\forall i \in I,\ \rho_i < n_i$.

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Application: process ergodic when π has finite mass. e.g. for $\phi_i(x) = \min(x, n_i)$, ergodicity iff $\forall i \in I,\ \rho_i < n_i$.
Proof: verify **partial balance** equations for all $x \in \mathbb{N}^I$:

$$\begin{array}{ll} \forall i \in I, & \\ \pi_x [\sum_{j \neq i} q_{x,x-e_i+e_j} + q_{x,x-e_i}] & = \sum_{j \neq i} \pi_{x-e_i+e_j} q_{x-e_i+e_j,x} + \pi_{x-e_i} q_{x-e_i,x} \\ \pi_x \sum_{i \in I} q_{x,x+e_i} & = \sum_{i \in I} \pi_{x+e_i} q_{x+e_i,x}, \end{array}$$

which imply global balance equations

$$\pi_x [\sum_{i \in I} (q_{x,x-e_i} + q_{x,x+e_i} + \sum_{j \neq i} q_{x,x-e_i+e_j})] =$$
$$\sum_{i \in I} (\pi_{x-e_i} q_{x-e_i,x} + \pi_{x+e_i} q_{x+e_i,x} + \sum_{j \neq i} \pi_{x-e_i+e_j} q_{x-e_i+e_j,x})$$

Takeaway messages

- Poisson process a fundamental continuous-time process, adequate model for aggregate of infrequent independent events

- Markov jump processes:
  i) generator Q characterizes distribution if not explosive
  ii) Balance equation characterizes invariant distribution if irreducible non-explosive
  iii) Limit theorems: stationary distribution reflects long-term performance

- Exactly solvable models include reversible processes, plus several other important classes (e.g. Jackson networks)