# MAP554 – Networks, PC9

November 29, 2016

## Push-Pull Gossip

Consider on a complete graph the following mechanism for disseminating information: Each node $i$, at the instants of its own Poisson $\lambda$ process contacts a uniformly selected target node $j$. If either $i$ or $j$ was infected then after this contact they both are.

**1.1** Determine the average time till total infection, when started from a single infected source. Show that the push-pull mechanism is twice as fast as the corresponding pull mechanism (standard SI process).

**1.2** Compare this push-pull mechanism to the *targeted* push mechanism where each node, at the instants of its Poisson process, infects one previously healthy node.

## 2. Computing averages with gossip

Nodes indexed by $[n]$ communicate over a network $G = (V, E)$. Each node $i$ initially holds a personal value $x_i(0) \in \mathbb{R}$. At (Poisson) rate $q_{ij} = q_{ji} = \mathbf{1}_{i \sim j}$, node $i$ contacts node $j$, after which they perform a local average of their personal value through the update

$$(x_i(t^+), x_j(t^+)) \leftarrow \left( \frac{x_i(t^-) + x_j(t^-)}{2}, \frac{x_i(t^-) + x_j(t^-)}{2} \right).$$

We will consider the symmetric matrix $L$ defined by $L_{ij} = -q_{ij}$, $i \neq j$ and $L_{ii} = \sum_{j \neq i} q_{ij}$, also known as the Laplacian of the graph.

**2.1** Show that $L$ is such that for each $x \in \mathbb{R}^n$, $x^T L x = \frac{1}{2} \sum_i \sum_{j \neq i} q_{ij} (x_i - x_j)^2$. We will call $\lambda_2$ the second smallest eigenvalue of $L$ and assume $\lambda_2 > 0$.

**2.2** For each $t > 0$, we consider the quantity $y(t) := \mathbb{E}||x(t) - \bar{x}e||^2$, where $\bar{x} = n^{-1} \sum_i x_i(0)$ and $e$ is the all-ones vector. Reasoning on the events that may occur during $[t, t+h]$, establish that

$$\frac{d}{dt} y(t) = -\mathbb{E}\left[ (x(t) - \bar{x}e)^T L (x(t) - \bar{x}e) \right].$$

**2.3** Deduce that $y(t) \leq y(0) e^{-\lambda_2 t}$, $t \geq 0$.

**2.4** Deduce that for all $\epsilon > 0$,

$$\mathbb{P}(\exists i \in [n] : |x_i(t) - \bar{x}| \geq \epsilon) \leq \frac{||x(0) - \bar{x}e||^2 e^{-\lambda_2 t}}{\epsilon^2}.$$

**2.5** Application: estimation of network size. Assume that initially, $x_1(0) = 1$ and $x_i(0) = 0$, $i \neq 1$. Determine as a function of a parameter $\delta$ a time $T$ such that for $t \geq T$, with probability at least $1 - \delta$, for each node $i$ one has $|x_i(t) - 1/n)| \leq 0.1/n$ (say).

The interpretation is that by that time, each node $i$ can form an estimate $\hat{n}_i := 1/x_i(t)$ of total system size $n$ that will be off by at most 10%.

# 3. Simple model of influence dynamics

Assume that $n$ individuals have a binary opinion, say -1 or $+1$. Each individual, at the instants of a unit rate Poisson process, succeeds in convincing an individual of previously opposite opinion to change her mind. Let $X(t)$ denote the number of individuals at time $t$ of opinion -1.

**3.1** Write recurrence equations on the quantity $\pi_x$ that conditional on $X(0) = x$, the $+1$ opinion prevails in the end.

**3.2** Solve these equations. How quickly does $\pi_x$ converge to zero or 1 as $x$ deviates from $n/2$?