MAP554 – Networks, PC7 with solutions

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1. $M/G/\infty/\infty$ queue and Poisson process

We consider a Poisson process $N \leftrightarrow \{T_n\}_{n>0}$ on $\mathbb{R}_+$ with intensity $\lambda > 0$ and to each point T_n attach a service time σ_n, where $\{\sigma_n\}_{n>0}$ is i.i.d., independent of N, with density f on $\mathbb{R}_+$.

1.1 Show that the set of points $\{(T_n, \sigma_n)\}_{n>0}$ constitutes a (generalized) Poisson process. Determine its intensity function.

1.2 The number X_t of customers present at time $t > 0$ is given by $X_t = \sum_{n>0} \mathbf{1}_{T_n \leq t < T_n+\sigma_n}$. Show that X_t admits a Poisson distribution. Determine its parameter as a function of λ and $\mathbb{E}(\sigma_1 \wedge t)$.

1.3 We now assume that N extends to all of $\mathbb{R}$ so that, letting $X_t = \sum_{n\in\mathbb{Z}} \mathbf{1}_{T_n \leq t < T_n+\sigma_n}$, we obtain a stationary process. Determine the stationary covariance $C(s) := \mathrm{Cov}(X_t, X_{t+s})$.

1.4 Is the process $\{X_t\}$ Markovian for general density f of service times σ_n?

2. $M/M/1/\infty$ queues with Processor Sharing discipline

We consider a single server queue with customer arrivals at instants of Poisson process N on $\mathbb{R}_+$ with intensity $\lambda > 0$, i.i.d. service times σ_n independent of N with Exponential(μ) distribution. Service discipline is **Processor Sharing**, i.e. when there are $k > 0$ customers present, each receives service at speed $1/k$.

2.1 Show that the number of customers in the queue is Markovian. Determine its transition rates and a stationary measure. Is the process reversible? Under what condition is it ergodic?

2.2 Assume now there are K distinct customer types, customers of type $i \in [K]$ arriving at instants of Poisson process N_i with intensity $\lambda_i > 0$, the N_i being mutually independent. Assume that service times of all customers of all types are i.i.d. with Exponential(μ) distribution (and independent of the N_i). Let $X_i(t)$ be the number of type i-customers present at time t. Answer same questions as in 2.1.

2.3 Assume now a network of L stations indexed by $\ell \in [L]$, K distinct customer types, $k \in [K]$. Assume a fixed network, with n_k customers of type k, each following a fixed cyclic route $\ell(1,k), \ell(2,k), \ldots, \ell(d_k,k), \ell(1,k), \ldots$, each ℓ appearing at most once per cycle. Finally assume that service at station ℓ is Processor Sharing, with service times there with Exponential(μ_ℓ) distribution.

Noting $X_{k\ell}$ the number of customers of type k at station ℓ, prove that a stationary measure for $\{X_{k\ell}\}_{k\in[K],\ell\in k}$ is given by, noting $y_\ell := \sum_{k\ni\ell} x_{k\ell}$,

$$\pi(x) = (\prod_{k\in[K]} \mathbf{1}_{\sum_{\ell\in k} x_{k\ell}=n_k}) \prod_{\ell\in[L]} \left(\frac{y_\ell! \mu_\ell^{-y_\ell}}{\prod_{k\ni\ell}(x_{k\ell}!)} \right)$$

Hint: Determine rates $q_{xx'}$ of generator, and associated rates $\tilde{q}_{xx'}$ such that

$$\pi(x) q_{xx'} = \pi(x') \tilde{q}_{x'x}, \ x \neq x', \tag{1}$$

then verify that $\sum_{x\neq x'} \tilde{q}_{x'x} = \sum_{x\neq x'} q_{x'x}$ to conclude.

2.4 We now set $\mu_\ell = AC_\ell$, $n_k = Aw_k$, for fixed w_k, C_ℓ, and let $A \to \infty$. We also set $x_{k\ell} = Av_{k\ell}$ and $y_\ell = Au_\ell$ with $u_\ell = \sum_{k\in\ell} v_{k\ell}$. Show with a crude version of Stirling's formula that for $A \to \infty$, the stationary distribution π concentrates its mass on solutions of the optimization problem

$$\begin{array}{ll} \text{Max} & \sum_{k\in[K]} \sum_{\ell\in k} v_{k\ell} \log(\frac{u_\ell}{C_\ell v_{k\ell}}) \\ \text{Over} & v_{k\ell} \geq 0, \ k \in [K], \ell \in k, \\ \text{such that} & \sum_{\ell\in k} v_{k\ell} = w_k, \ k \in [K]. \end{array} \tag{2}$$

2.5 The above scenario admits the following interpretation: service types correspond to individual transmissions; each transmission is regulated by a *fixed window control* with Aw_k the window size in number of packets, and C_ℓ the capacity (in bytes/s) of server ℓ. The limit $A \to \infty$ corresponds to a "small data packets / high transmission rates" regime.

We will admit that (**??**) is a concave maximization problem, whose optimum $\{v^*_{k\ell}\}$ is characterized as achieving the maximum of

$$L(\{v_{k\ell}\}, \{\beta_k\}) := \sum_{k\in[K]} \sum_{\ell\in k} v_{k\ell} \log(\frac{u_\ell}{C_\ell v_{k\ell}}) + \sum_{k\in[K]} \beta_k (w_k - \sum_{\ell\in k} v_{k\ell})$$

over $\{v_{k\ell}\} \geq 0$ for some suitable vector of multipliers $\{\beta_k\} \in \mathbb{R}^K$.

Argue from the corresponding solution, taking $C_\ell v_{k\ell}/u_\ell$ as the rate of transmission k for any $\ell \in k$ with $u_\ell > 0$, that the resulting rates correspond to $(w, 1)$-fairness, or weighted proportional fairness.

3. Jackson networks and Kleinrock's square root law

Consider a Jackson network with stations $i \in I$, routing probabilities p_{ij}, single server queues at each station, and service time distributions Exponential(1). Let $\lambda_i > 0$ be the solutions of the traffic equations. Assume that a total capacity C is available, and to be distributed among the servers.

3.1 Write the stationary measure for a particular allocation C_i of capacity to each server i, $C_i > 0$, $\sum_{i\in I} C_i = C$.

3.2 Determine under which condition an allocation C_i makes the system ergodic.

3.3 Assume the system can be made ergodic. Determine the allocation which minimizes the average number of customers in the system.