MAP554 – Networks: PC 2

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1. Birkhoff-von Neumann theorem

A square matrix $M \in \mathcal{M}_n(\mathbb{R})$ is *doubly stochastic* if for all $i, j \in [n]$, $M_{ij} \geq 0$ and

$$M_{i\cdot} := \sum_{j' \in [n]} M_{ij'} = 1, \quad M_{\cdot j} := \sum_{i' \in [n]} M_{i'j} = 1.$$

Denoting by $\mathcal{S}_n$ the symmetric group of permutations of $[n]$, for any $\sigma \in \mathcal{S}_n$ we define the associated permutation matrix as $M_\sigma := (\mathbf{1}_{\sigma(i)=j})_{i,j \in [n]}$. The theorem states that the set $\mathcal{A}$ of doubly stochastic matrices coincides with the convex hull $\mathcal{B}$ of permutation matrices.

1.1 Show that $\mathcal{B} \subset \mathcal{A}$.

1.2 Let M be a doubly stochastic matrix M. For any set $R \subset [n]$ of rows, let $\mathcal{N}(R)$ be the set of columns $\{c : \exists r \in R : M_{rc} > 0\}$. Show that for all sets of rows R, $|R| \leq |\mathcal{N}(R)|$.

1.3 Hall's marriage theorem: Show that for a bipartite graph with bipartition of vertices into two finite sets $\mathcal{R}$, $\mathcal{C}$, there is an injection $\sigma : \mathcal{R} \to \mathcal{C}$ such that for all $r \in \mathcal{R}$, $(r, \sigma(r))$ is an edge of the graph if and only if for all $R \subset \mathcal{R}$, $|\mathcal{N}(R)| \geq |R|$.

Hint: without loss of generality take $\mathcal{R} = [n]$, and use induction on $n := |\mathcal{R}|$. Distinguish according to whether for all $k < n$, and all $R \subset \mathcal{R}$ with $|R| = k$, $|\mathcal{N}(R)| \geq k + 1$ (in which case use induction to match the first $n - 1$ elements of $\mathcal{R}$ to $\mathcal{C} \setminus \{j\}$ for some $j \in \mathcal{C}$ neighbour of $n \in \mathcal{R}$), or whether there exists $k < n$ and $R \subset \mathcal{R}$ with $|R| = |\mathcal{N}(R)| = k$.

1.4 Deduce the result of the Birkhoff-von Neumann theorem from 1.2) and 1.3).

1.5 One says that a matrix M is doubly sub-stochastic if for all $i, j \in [n]$, $M_{ij} \geq 0$ and $M_{i\cdot} := \sum_{j' \in [n]} M_{ij'} \leq 1$, $M_{\cdot j} := \sum_{i' \in [n]} M_{i'j} \leq 1$.

Show that a matrix M is doubly sub-stochastic if and only if there exists a doubly stochastic matrix N such that for all $i, j \in [n]$, $M_{ij} \leq N_{ij}$.

2. Maximal (and not maximum weight) matching for crossbar switches

Consider an i.i.d. sequence $\{A_n\}_{n \in \mathbb{N}}$ where $A_n \in \mathbb{N}^{d \times d}$ and $A_n(i, j)$ represents the numbers of packet arrivals in time slot n to be routed from some input port $i \in [d]$ to some output port $j \in [d]$. Let $\lambda_{ij} := \mathbb{E} A_n(i, j)$ denote its mean. Assume that $\mathbb{P}(\forall i, j \in [d],\ A_n(i, j) = 0) \in]0, 1[$.

Let $Q_n(i,j)$ denote the number of enqueued packets at input port i destined to output port j at the beginning of time slot n. We know that a maximum weight selection of a permutation σ_n, maximizing $\sum_{i\in[d]} Q_n(i,\sigma(i))$ renders the Markov process $\{Q_n\}_{n\in\mathbb{N}}$ ergodic when for some $\epsilon>0$, $((1+\epsilon)\lambda_{ij})_{i,j\in[d]}$ is doubly sub-stochastic. However it is costly to determine a maximum-weight matching (best known complexity: $O(d^3)$).

We instead consider using a *maximal* matching defined as follows. A matching of a graph is defined as a collection of graph edges such that no two distinct edges are adjacent to a common vertex. Such a matching is called maximal if no edge of the graph can be added to it without breaking the matching property.

We then consider the graph with input and output ports as vertices, and as set of edges the pairs $(i,j)\in[d]^2$ such that $Q_n(i,j)>0$. A maximal matching algorithm schedules transmissions of enqueued packets along edges of a maximal matching. It can be implemented fast (in $O(d\ln(d))$ operations).

2.1 Let $M_n(i,j)\in\{0,1\}$ indicate whether edge (i,j) is picked or not in the maximal matching used for scheduling in time slot n. Show that, if $Q_n(i,j)>0$, then

$$\sum_{i'\in[d]} M_{i'j}(n)+\sum_{j'\in[d]} M_{ij'}(n)\geq 1.$$

2.2 Show, using Lyapunov function $V(q):=\sum_{i,j\in[d]} q_{ij}\left[\sum_{i'\in[d]} q_{i'j}+\sum_{j'\in[d]} q_{ij'}\right]$, that process $\{Q(n)\}$ is ergodic if for some $\epsilon>0$, matrix $(2(1+\epsilon)\lambda_{i,j})_{i,j\in[d]}$ is doubly sub-stochastic and for all i,j, $\mathbb{E}(A_{ij}^2)<+\infty$.

Hint: establish that for all i,j, one has $\sum_{i',j'}\lambda_{i'j}+\lambda_{ij'}\leq 1/(1+\epsilon)$.

3. Load-balanced switch

Assume now that the $A_{ij}(n)$ are Bernoulli random variables, i.e. $A_{ij}(n)\in\{0,1\}$. Consider the switching architecture consisting of two switching stages. In the first stage, during time slot n, a packet arrived at input port i and due to output port j is forwarded to input port $i+n \bmod d$ of the second stage, to contribute to the input queue indexed by $(i+n \bmod d, j)$. During the same time slot, in the second switching stage, the same permutation as in the first stage, mapping i to $i+n \bmod d$, is used to schedule transmissions.

3.1 Argue that the input queues of the first stage are always empty.

3.2 Compute the mean arrival rate μ_{ij} of packets to the input port i of the second stage switch, that are destined to its output port j.

3.3 Argue that this architecture ensures ergodicity provided for some $\epsilon>0$, matrix $((1+\epsilon)\lambda_{ij})$ is doubly sub-stochastic. Explain how to represent the system's evolution as a Markov chain with transition probabilities that do not depend on time slot n (even modulo d).