MAP554 – Networks: PC 1

14 September 2016

1. Recurrence / transience of reflected random walks

For i.i.d. random variables $\{Y_k\}_{k\in\mathbb{N}} \in \mathbb{Z}$ consider the reflected random walk $\{X_k\} \in \mathbb{N}$ defined via $X_{k+1} = \max(0, X_k + Y_{k+1})$.

1.1 Show that the chain is irreducible on $\mathbb{N}$ if $\mathbb{P}(Y_k < 0) > 0$ and $\mathbb{P}(Y_k = 1) > 0$.

1.2 Show that when $\mathbb{E}(Y_k) > 0$ the chain is transient, and $\mathbb{P}(\exists n > 0 : X_n = 0 | X_0 = x)$ goes to zero as $x \to +\infty$.

1.3 Show that when $\mathbb{E}(Y_k) < 0$ the chain is ergodic using the Foster-Lyapunov criterion.

1.4 Prove that for i.i.d. $\{Y_k\}$, noting $\mathcal{F}_t = \sigma(Y_1, \dots, Y_t)$, for an almost surely bounded $\mathcal{F}_n$-stopping time N, $\mathbb{E}\sum_{k=1}^{N} Y_k = \mathbb{E}(N)\mathbb{E}(Y_1)$ (this is known as Wald's identity).

1.5 Use 1.4) to re-derive the result of 1.3) in the case where Y_k is bounded from below ($\exists m \in \mathbb{N} : \mathbb{P}(Y_k \geq -m) = 1$), considering the bounded stopping times $N \wedge n$ for $N := \inf\{k \geq 0 : X_k = 0\}$.

1.6* Assume $\mathbb{E}(Y_k) = 0$ and $0 < \mathrm{Var}(Y_k) = \sigma^2 < +\infty$. Show that the chain is null recurrent.

Hints: it cannot be positive recurrent (argue by contradiction assuming a stationary regime $\{X_k, Y_k\}$, showing with the help of Bienaymé-Tchebitchev inequality that necessarily $\forall k \in \mathbb{N}$, $\Delta_k = X_{k+1} - X_k - Y_{k+1} \equiv 0$, hence $X_k = X_0 + Y_1 + \dots Y_k$, then using the Central Limit Theorem, or CLT to reach a contradiction).

It is recurrent: consider successive stopping times $T(k)$: $T(k+1) = T(k) + \max(m, X^2_{T(k)})$ for large enough, fixed $m \in \mathbb{N}$. Use the CLT show that for some positive ϵ, $\mathbb{P}(X_t \text{ hits } 0 \text{ in } t \in [T(k), T(k+1)] | \mathcal{F}_{T(k)}) \geq \epsilon$.

2. Instability of Aloha

The i.i.d. sequence $\{A_n\} \in \mathbb{N}$ denotes the number of messages arriving in time slots $n \in \mathbb{N}$. The doubly infinite array $\{B_{n,i}\}_{n,i\in\mathbb{N}} \in \{0,1\}$ of i.i.d., mean p, Bernoulli random variables, independent of the $\{A_n\}$, specifies transmission attempts. The number of queued messages L_n at the beginning of slot n satisfies:

$$L_{n+1} = L_n + A_n - \mathbf{1}_{\sum_{i=1}^{L_n} B_{n,i}=1}.$$

2.1 Show irreducibility of the chain $\{L_n\}$ when $0 < \mathbb{P}(A_n = 0) < 1$, a condition assumed to hold from now on.

2.2 Consider the chain $\hat{L}_n$ defined by $\hat{L}_0 = L_0 = \ell$, $\hat{L}_{n+1} = \hat{L}_n + A_n$, and the binary variables $\hat{Z}_n = \mathbf{1}_{\sum_{i=1}^{\hat{L}_n} B_{n,i} \geq 2}$, $Z_n = \mathbf{1}_{\sum_{i=1}^{L_n} B_{n,i} \geq 2}$.

Show that on the event $\mathcal{E} := \{\forall n \in \mathbb{N}, \hat{Z}_n = 1\}$ one has $\forall n \in \mathbb{N}, \hat{L}_n = L_n$.

2.3 Show that for large enough ℓ the event $\mathcal{E}$ has positive probability (say at least 1/4), by upper-bounding the expectation $\mathbb{E}[\sum_{n\in\mathbb{N}}(1-\hat{Z}_n)|\hat{L}_0 = \ell, A_0^\infty]$.

Hint: consider the random variable $Y := \sup_{n\in\mathcal{N}}[n(a/2) - (A_0 + \ldots + A_{n-1})]$ where $a := \mathbb{E}(A_k) > 0$, show that it is almost surely finite, and establish the inequality

$$\hat{L}_n \geq n(a/2) - Y + \ell.$$

2.4 Deduce transience of $\{L_n\}$.

2.5 Show that with probability 1, only finitely many packets are transmitted by Aloha.

Hint: Consider the Markov chain $\{(L_n, Z_n)\}$ and the corresponding sequence of stopping times $T_k < T_{k+1} \cdots$ corresponding to the instants n where $Z_n = 0$. Show that the probability that $T_{k+1} < +\infty$ conditionally on $T_k < +\infty$ is no larger than $1 - \epsilon$ for some positive ϵ.

3. variant of Aloha with shared information

Here again $\{A_n\}$ is i.i.d. and denotes the number of new packet arrivals during slot n. We assume moreover that A_n has Poisson distribution with parameter $\lambda > 0$. Finally we suppose that all packet transmitters share information under the form of a clock, i.e. they all know the value n of the current time slot.

3.1 Suggest a variant of Aloha, which exploits value of current time n, and succeeds in transmitting at rate at least $\mathbb{P}(A_1 = 1)/2 = (\lambda e^{-\lambda})/2$, i.e. noting L_n the number of backlogged packets at the beginning of slot n, and D_n the number of packets departing during slot n, the scheme would ensure that almost surely

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} D_k \geq (\lambda e^{-\lambda})/2.$$

3.2 Can you achieve a success rate of $\lambda e^{-2\lambda}$?

4. Self-organizing lists

Consider a memory storing D documents as an ordered list, $\sigma_i \in [D]$ being the identity of the i-th document in the list. The cost of requesting a document is assumed to equal its rank in the list. At each time unit, a document is requested, whose identity is drawn i.i.d. according to the distribution $\{\pi_1, \ldots, \pi_D\}$. We let R_n denote the document requested at time n.

4.1 Characterize for a fixed order $\sigma_1, \ldots, \sigma_D$, the average cost per time unit of serving requests R_n, $n \in \mathbb{N}$. Determine the static ordering $\sigma_1, \ldots, \sigma_D$ which minimizes this cost.

4.2 Consider the "Move-To-Front" rule: after serving request for document R_n the list σ is updated by moving the document R_n to the front of the list, leaving the order of other documents otherwise unchanged. Show that this defines a Markov chain on the symmetric group of permutations of D elements. Under what condition is it irreducible?

4.3 Characterize the stationary distribution of this chain.

4.4 Prove that the probability that element i appears before element j under the stationary distribution of the Move-To-Front permutation is given by $\pi_i/(\pi_i + \pi_j)$.

4.5 Deduce that the average search cost under Move-To-Front reads

$$1 + 2 \sum_{1 \le i < j \le D} \frac{\pi_i \pi_j}{\pi_i + \pi_j}.$$

4.6 Deduce that the average search cost under Move-To-Front is no larger than twice the average search cost under the best static list ordering.